J. Appl. Maths Mechs Vol. 56, No. 5, pp. 623–629, 1992
Printed in Great Britain.

0021–8928/92 $24.00 + .00

UNCOUPLED DUAL FORMULATIONS OF THE VARIATIONAL BOUNDARY ELEMENT METHOD IN PROBLEMS OF THE THEORY OF ELASTICITY†

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(*Received* 27 *November* 1991)

Uncoupled dual formulations (UDFs), different from those considered previously [1–3], are proposed for the boundary functionals of the linear theory of elasticity, in the sense that the displacements and stresses are varied independently, and the equations of state on the boundary are treated as constraints involving Lagrange multipliers. The idea of this device—using Lagrange multipliers to get rid of restrictions in the variational problem, represented by the equations of state—was used previously [4] to formulate dual variational problems of the linear theory of elasticity based on the Lagrange–Castigliano principle. A finite element approximation of the solutions of these problems yields mixed formulations of the finite element method [5]. Thus, the boundary element approximations (BEAs) proposed below for the UDF may be regarded as a special mixed finite element method. Simultaneous BEA of the displacements and stresses extends the applicability of UDFs to cases in which allowance must be made for singularities of the solution, e.g. in contact problems of the theory of elasticity and in fracture mechanics (crack problems).

1. LET $G\subset E_m$ $(m = 2, 3)$ be a region in a (possibly infinite) elastic medium with sufficiently smooth boundary S. The problem of minimizing the quadratic energy functional for the second boundary-value problem of linear elasticity theory (with given stresses on S) may be reduced [6] to the equivalent problem of minimizing a boundary functional over the kinematically admissible displacements

$$\min_{\mathbf{u}\in D} F_S(\mathbf{u}),\quad F_S=\int_S \mathbf{u}\mathbf{t}^{(\nu)}(\mathbf{u})ds-2\int_S \mathbf{u}\mathbf{g}^{(\nu)}ds$$
$$D=\{\mathbf{u}|\ \mathbf{A}\mathbf{u}(x)=0,\ x\in G\}\tag{1.1}$$

Body forces are not taken into consideration; $\mathbf{g}^{(\nu)}(y)$, $y\in S$, is the vector of given normal stresses on S; the set D of displacement vectors that satisfy the equation of equilibrium of the elastic medium is the set of linear constraints of the variational problem. A solution of problem (1.1) exists, apart from an arbitrary rigid displacement. Under conditions that exclude such a displacement we have [6]

$$\min_{\mathbf{u}\in D} F_S(\mathbf{u})=F_S(\mathbf{u}_0)=d_0=-\int_S \mathbf{u}_0\mathbf{t}^{(\nu)}(\mathbf{u}_0)ds$$

where $\mathbf{u}_0$ is the vector of elastic displacements—the solution of the second problem of the theory of elasticity. Problem (1.1) is "coupled" in the sense that the variables—the displacement vector $\mathbf{u}$ and the stress vector on the boundary $\mathbf{t}^{(\nu)}$—must satisfy the defining relations

$$\mathbf{t}^{(\nu)}(\mathbf{u})=\sum_{i,k,l,r} c_{iklr}(x)\epsilon_{lr}\cos(\nu,x_i)\,\mathbf{x}_k^{(0)}\tag{1.2}$$

We will henceforth study an "uncoupled" formulation, with these relationships maintained valid

† *Prikl. Mat. Mekh.* Vol. 56, No. 5, pp. 729–736, 1992.

as constraint equations through the use of Lagrange multipliers. We shall first prove a duality relation (not the same as that previously established [6] for coupled formulations).

Let T be the set of boundary values of the statically admissible stress vectors. We define Lagrange multipliers as sufficiently smooth vector functions λ defined at the points of S and such that the real-valued function

$$f(\mathbf{u}, \mathbf{t}^{(\nu)}, \lambda) = \int_{S'} \lambda[\mathbf{t}^{(\nu)} - \mathbf{t}^{(\nu)}(\mathbf{u})]\,ds, \quad \mathbf{u} \in D, \quad \mathbf{t}^{(\nu)} \in T \tag{1.3}$$

satisfies the equality

$$\sup_\lambda f(\mathbf{u}, \mathbf{t}^{(\nu)}, \lambda) = \begin{cases} 0, & \mathbf{t}^{(\nu)} = \mathbf{t}^{(\nu)}(\mathbf{u}) \\ +\infty, & \mathbf{t}^{(\nu)} \neq \mathbf{t}^{(\nu)}(\mathbf{u}) \end{cases}$$

This is established by a direct check, and the function f is used to construct the Lagrangian

$$\begin{gathered} L(\mathbf{u}, \mathbf{t}^{(\nu)}, \lambda) = w(\mathbf{u}, \mathbf{t}^{(\nu)}) - 2l(\mathbf{u}) - f(\mathbf{u}, \mathbf{t}^{(\nu)}, \lambda) \\ w = \int_S \mathbf{u}\mathbf{t}^{(\nu)}ds, \quad l = \int_S \mathbf{u}\mathbf{g}^{(\nu)}ds \end{gathered} \tag{1.4}$$

(later we shall suggest an interpretation of the multipliers λ). It can be shown that the problem of finding a saddle point $\{\mathbf{u}_0, \mathbf{t}^{(\nu)}(u_0), \lambda_0\}$ of L, that is, the direct problem

$$\inf_{\mathbf{u}, \mathbf{t}^{(\nu)}} \sup_\lambda L$$

is equivalent to the variational problem (1.1). Indeed, if $\mathbf{t}^{(\nu)} = \mathbf{t}^{(\nu)}(\mathbf{u})$, then $\sup_\lambda f = 0$ and

$$\inf_{\mathbf{u}, \mathbf{t}^{(\nu)}(\mathbf{u})} \sup_\lambda L(\mathbf{u}, \mathbf{t}^{(\nu)}(\mathbf{u}), \lambda) = \inf_{\mathbf{u}} F_S(\mathbf{u}) = d_0 \tag{1.5}$$

The dual problem, to determine

$$\sup_\lambda \inf_{\mathbf{u}, \mathbf{t}^{(\nu)}} L \tag{1.6}$$

is meaningful if the duality relation holds

$$\inf_{\mathbf{u}, \mathbf{t}^{(\nu)}} \sup_\lambda L = \sup_\lambda \inf_{\mathbf{u}, \mathbf{t}^{(\nu)}} L = d_0. \tag{1.7}$$

Thus, we have to prove the right-hand side of (1.7). For fixed λ, the solution $(\mathbf{u}_\lambda, \mathbf{t}_\lambda^{(\nu)})$ of the problem of finding $\inf_{\mathbf{u}, \mathbf{t}^{(\nu)}} L$ is determined from the system of equations

$$\begin{gathered} \operatorname{grad}_{\mathbf{u}} L(\mathbf{u}_\lambda, \mathbf{t}_\lambda^{(\nu)}, \lambda) = 0 \\ \operatorname{grad}_{\mathbf{t}^{(\nu)}} L(u_\lambda, t_\lambda^{(\nu)}, \lambda) = 0 \end{gathered}$$

which, in view of (1.4), can be written in the form

$$w(\mathbf{v}, \mathbf{t}_\lambda^{(\nu)}) - 2l(\mathbf{v}) + \int_S \lambda \mathbf{t}^{(\nu)}(\mathbf{v})ds = 0, \quad \forall\, \mathbf{v} \in D \tag{1.8}$$

$$w(\mathbf{u}_\lambda, \tau^{(\nu)}) - \int_S \lambda \tau^{(\nu)} ds = 0, \quad \forall\, \tau^{(\nu)} \in T \tag{1.9}$$

Under these conditions the value of L for the solution $(\mathbf{u}_\lambda, \mathbf{t}_\lambda^{(\nu)})$ is obtained from the explicit expression for L and from (1.8) with $\mathbf{v} = \mathbf{u}_\lambda$ and (1.9) with $\tau^{(\nu)} = \mathbf{t}_\lambda^{(\nu)}$

$$\begin{aligned} L(\mathbf{u}_\lambda, \mathbf{t}_\lambda^{(\nu)}, \lambda) &= -\int_S \lambda \mathbf{t}^{(\nu)}(\mathbf{u}_\lambda)ds - \int_S \lambda \mathbf{t}_\lambda^{(\nu)} ds + \int_S \lambda \mathbf{t}^{(\nu)}(\mathbf{u}_\lambda)ds = \\ &= -\int_S \lambda \mathbf{t}_\lambda^{(\nu)} ds = -w(\mathbf{u}_\lambda, \mathbf{t}_\lambda^{(\nu)}) = \inf_{\mathbf{u}, \mathbf{t}^{(\nu)}} L \end{aligned}$$

Then the dual problem (1.6) reduces to

$$\sup_\lambda[-w(\mathbf{u}_\lambda, \mathbf{t}_\lambda^{(\nu)})] = -\inf_\lambda w(\mathbf{u}_\lambda, \mathbf{t}_\lambda^{(\nu)})$$

where $w(\mathbf{u}_\lambda, \mathbf{t}_\lambda^{(\nu)})$ is a quadratic form in λ. Next, we deduce from the condition

$$\operatorname{grad}_\lambda L(\mathbf{u}, \mathbf{t}^{(\nu)}, \lambda) = \int_S \delta[\mathbf{t}^{(\nu)} - \mathbf{t}^{(\nu)}(\mathbf{u})]ds = 0 \quad \forall \delta \in D \tag{1.10}$$

that $\mathbf{t}_\lambda^{(\nu)} = \mathbf{t}^{(\nu)}(\mathbf{u}_\lambda)$ for every fixed λ. Hence it follows that for $\lambda = \lambda_0$ and $\mathbf{u}_{\lambda_0} = \mathbf{u}_0$, $\mathbf{t}^{(\nu)}(\mathbf{u}_{\lambda_0}) = \mathbf{t}^{(\nu)}(\mathbf{u}_0)$

$$-\inf_\lambda w(\mathbf{u}_\lambda, \mathbf{t}_\lambda^{(\nu)}) = -w(\mathbf{u}_0, \mathbf{t}^{(\nu)}(\mathbf{u}_0)) = d_0$$

These results, together with (1.5), prove (1.7).

Relationships (1.9) and (1.10) may be considered, first, as integral identities, in which case the first implies an interpretation of the Lagrange multiplier $\lambda_0 \equiv \mathbf{u}_0$—the vector of elastic displacements; the second relationship implies the satisfaction of the coupling equations $\mathbf{t}^{(\nu)} = \mathbf{t}^{(\nu)}(\mathbf{u}_0)$. Alternatively, the same relationships may be regarded as variational equations which, together with equation (1.8), may be used to construct BEAs of the solution of the dual problem (1.6). By what we have already proved, the infima $\inf_{\mathbf{u},\mathbf{t}^{(\nu)}}$, $\sup_\lambda$ in (1.7) are attained; hence they may be written instead as $\min_{\mathbf{u},\mathbf{t}^{(\nu)}}$, $\max_\lambda$. The minimum smoothness conditions for the variables of these dual formulations are essentially membership conditions: $\mathbf{u}, \lambda \in W_2^{1/2}(S)$, $\mathbf{t}^{(\nu)} \in W_2^{-1/2}(S)$, where $W_2^{1/2}(S)$ is the appropriate Sobolev–Slobodetskii space and $W_2^{-1/2}(S)$ is its dual.

2. The saddle point $\{\mathbf{u}_0, \mathbf{t}^{(\nu)}(\mathbf{u}_0), \lambda_0\}$ of the Lagrangian L, whose existence follows from (1.7), is characterized by the condition [7]

$$\begin{aligned} &L(\mathbf{u}_0, \mathbf{t}^{(\nu)}(\mathbf{u}_0), \lambda) \leqslant L(\mathbf{u}_0, \mathbf{t}^{(\nu)}(\mathbf{u}_0), \lambda_0) = \\ &= d_0 \leqslant L(\mathbf{u}, \mathbf{t}^{(\nu)}, \lambda_0), \quad \forall \mathbf{u}, \mathbf{t}^{(\nu)}, \lambda \end{aligned} \tag{2.1}$$

To satisfy (2.1) we must solve the system of variational equations (1.8)–(1.10); there are different possible kinds of BEA: isoparametric, superparametric and subparametric [8].

From now on we shall consider a homogeneous isotropic medium occupying the region G. Suppose we have isoparametric approximations of the boundary and of the variables $\mathbf{u}$, $\mathbf{t}^{(\nu)}$, λ at the points of a boundary element (BE) Δs_n (the boundary S is divided into such elements)

$$\begin{aligned} &y_n = \sum_{i,k} Y_{nk}^{(i)} \psi_k, \quad \mathbf{u}_n = \sum_{i,k} U_{nk}^{(i)} \psi_k \\ &\mathbf{t}_n^{(\nu_n)} = \sum_{i,k} T_{nk}^{(i)} \psi_k, \quad \lambda_n = \sum_{i,k} \Lambda_{nk}^{(i)} \psi_k \end{aligned} \tag{2.2}$$

Here $Y_{nk}^{(i)}$ are the Cartesian coordinates of the nodes k of the discrete boundary $S_\Delta = \cup \Delta s_n$ ($n = 1, \ldots, N$), and $\mathbf{U}_{nk}$, $\mathbf{T}_{nk}$, Λ_{nk} are the nodal values of the respective variables; $\psi_k(\eta)$ are the basis functions of the BEM [8] and η is the local coordinate of the points of Δs_n. Summation in (2.2) is performed from $i = 1$ to $i = m$ and from $k = 1$ to $k = K$.

Global interpolating functions $\mathbf{u}_N(y_\Delta)$, $\mathbf{t}_N^{(\nu_\Delta)}(y_\Delta)$, $\lambda_N(y_\Delta)$ at points $y_\Delta \in S_\Delta$ are obtained by summing (2.2) over $n = 1, \ldots, N$. As the nodal values are equal at common nodes of adjacent elements (by the compatibility condition for BEs [8]), the functions thus obtained are continuous at the points of S_Δ. Functions approximating the solution at the points of the region G_Δ bounded by S_Δ may be derived from the boundary values $\mathbf{u}_N . \mathbf{t}_N^{(\nu_\Delta)}$ as superpositions of boundary potentials ([1, 2], see also [9])

$$\begin{aligned} \alpha_N(x_\Delta) = &-\frac{1}{2} \int_{S_\Delta} \mathbf{t}^{(\nu_\Delta)}\Big(\sum_{j=1}^{m} \mathbf{v}^{1j}\Big) \mathbf{u}_N(y_\Delta) ds(y_\Delta) + \\ &+ \frac{1}{2} \int_{S_\Delta} \sum_{j=1}^{m} \mathbf{v}^{1j} \mathbf{t}_N^{(\nu_\Delta)}(y_\Delta) ds(y_\Delta), \quad x_\Delta \in G_\Delta \end{aligned} \tag{2.3}$$

where $\{v^{ij}\}$ $i, j = 1, \ldots, m$ is the tensor of fundamental solutions of the Lamé equations (the Somigliana tensor [10]).

The representation (2.3), which is known [11] to be rigorous for piecewise-smooth boundaries S_Δ, associates the same vector function with uncoupled boundary values $\mathbf{u}_N$, $\mathbf{t}_N^{(\nu_\Delta)}$: the solution of the Lamé equation in the region inside (and outside) S_Δ. Thus, the functions

$$\alpha_N \in D_\Delta = \{x_\Delta |\ \mathrm{A}\alpha_N(x_\Delta) = 0,\quad x_\Delta \in G_\Delta\}$$

are admissible functions of the finite-dimensional variational problem for the discrete Lagrangian $L_\Delta(\mathbf{u}_N, \mathbf{t}_N^{(\nu_\Delta)}, \lambda_N)$. Applied to the approximations $\{\mathbf{u}_N\}$, $\{\mathbf{t}_N^{(\nu_\Delta)}\}$, $\{\lambda_N\}$, the variational equations (1.8)–(1.10) reduce [1, 2] to the following system of discrete boundary equations

$$\begin{aligned}
&\sum_{n=1}^{N} \left[\int \mathbf{t}_n^{(\nu_n)}\psi_l |J_n|\, ds_n + \int \lambda_n c \frac{\partial \psi_l}{\partial \nu_n} |J_n|\, ds_n\right] = \\
&= 2 \sum_{n=1}^{N} \int \mathbf{g}_n^{(\nu_n)} \psi_l |J_n|\, ds_n, \quad l = 1, \ldots, K \\
&\sum_{n=1}^{N} \left[\int \mathbf{u}_n \psi_l |J_n|\, ds_n - \int \lambda_n \psi_l |J_n|\, ds_n\right] = 0 \\
&\sum_{n=1}^{N} \left[\int \mathbf{t}_n^{(\nu_n)} \psi_l |J_n|\, ds_n - \int \mathbf{t}^{(\nu_n)}(\mathbf{u}_n)\psi_l |J_n|\, ds_n\right] = 0
\end{aligned} \tag{2.4}$$

The integration here is performed over the union of the BEs Δs_n for which the node k [see (2.2)] is common; we have used the following notation (see also [9]): $|J_n|$ is the determinant of the Jacobian of the transformation of a surface element $ds_n(\eta)$ in local coordinates to a surface element $ds_n(y)$ in global (Cartesian) coordinates; $\mathbf{g}_n^{(\nu_n)}$ is a BEA (2.2) for the given vector $\mathbf{g}^{(\nu)}$ [see (1.1)], ν_n is the outward normal at points of Δs_n; $c(\theta, \mu)$ is a constant, which depends on the Lamé constants, which appears [9] on passing from the approximation $\mathbf{u}_n$ [see (2.2)] to the stress vector approximation $\mathbf{t}^{(\nu)}(\mathbf{u}_n)$. In simple cases, such as twisting of a homogeneous elastic rod, we have [9] $\mathbf{t}^{(\nu)}(\mathbf{u}_n) = 2\mu^{\partial \mathbf{u}_n}/\partial\nu$; in the general case the operator $c\partial/\partial\nu$ is equivalent to a certain scalar operator, whose actual form is determined [9] by the vector $\mathbf{t}^{(\nu)}(\mathbf{u}_n)$ [see (1.2)].

Let us compare the approach proposed here with the Ritz BEA of coupled variational formulations [1, 2, 9]. Unlike the Ritz procedure, where the gradient of a discrete functional corresponds to differentiating a quadratic form in the parameters (the unknown nodal values), here the gradient corresponds to differentiation with respect to parameters that appear linearly in the discrete Lagrangian L_Δ. Thus, the first equation in (2.4) corresponds to $\mathrm{grad}_{\mathbf{U}_{nk}} L_\Delta = 0$ ($\mathbf{U}_{nk}$ is a vector) and is expressed in terms of the parameters $\mathbf{T}_{nk}$, Λ_{nk}; the second and third equations correspond to $\mathrm{grad}_{\mathbf{T}_{nk}} L_\Delta = 0$ and $\mathrm{grad}_{\Lambda_{nk}} L_\Delta = 0$ and are expressed in terms of the parameters $\mathbf{U}_{nk}$, Λ_{nk} and $\mathbf{T}_{nk}$, $\mathbf{U}_{nk}$, respectively. It follows that the variational equations (1.8)–(1.10) may also be regarded as equations for the construction of a BEA "à la Galerkin" (see [5]).

We shall now show that system (2.4) is identical with the system of discrete boundary equations of the BEA in the case of problem (1.1), when evaluated for the Ritz approximations $\{\mathbf{u}_N\}$ [1, 2, 9]. The analysis runs as follows. Using the third equation of system (2.4), eliminate the first term from the first equation. The resulting system of two equations is fairly easy to analyse: the second equation implies $\mathbf{u}_N \equiv \lambda_N$, in agreement with the established fact that the Lagrange multipliers are simply the displacement vectors. Considering the BEA

$$\mathbf{t}^{(\nu_n)}(\mathbf{u}_n) = \sum_{i=1}^{m} \sum_{k=1}^{K} U_{nk}^{(i)} c(\theta, \mu) \frac{\partial \psi_k}{\partial \nu_n}$$

the symmetry condition satisfied by the matrix coefficients of the BEM [1, 9]

$$\int_{\Delta s_n} \frac{\partial \psi_k}{\partial \nu_n} \psi_l |J_n|\, ds_n(\eta) = \int_{\Delta s_n} \psi_k \frac{\partial \psi_l}{\partial \nu_n} |J_n|\, ds_n(\eta), \quad k, l = 1, \ldots, K$$

and also the equality $\mathbf{U}_{nk} = \Lambda_{nk}$ (as a corollary $\mathbf{u}_n \equiv \lambda_n$), we deduce from the first equation that

$$2\sum_{n=1}^{N}\int_{U\Delta s_n}\sum_{i=1}^{m}\sum_{k=1}^{K}U_{nk}^{(i)}c(\theta,\mu)\frac{\partial\psi_k}{\partial\nu_n}\psi_l|J_n|\,ds_n(\eta)=$$

$$=2\sum_{n=1}^{N}\int_{U\Delta s_n}\sum_{i=1}^{m}\sum_{k=1}^{K}Q_{nk}^{(i)}\psi_k\psi_l|J_n|\,ds_n(\eta),\quad l=1,\ldots,K \tag{2.5}$$

which is just the system of discrete boundary equations for the BEA of problem (1.1).

In connection with this uncoupled variational formulation for boundary functionals of the theory of elasticity, it should be mentioned that in the linear theory of elasticity one also has occasion to deal with functionals in which the displacement and stress variables may be varied independently, such as the Hellinger–Reissner functional. Variational formulations for this functional have been used as a basis [5] for mixed finite-element approximations of solutions to dual problems. Uncoupled formulations and an appropriate duality algorithm for their implementation may also be based on minimizing generalized Trefftz functionals of boundary-value problems of the linear theory of elasticity. Such an algorithm and a variational-difference scheme for its implementation may be found in [12].

3. We will now proceed to justify the algorithms for constructing BEAs. It can be proved that if $S_\Delta\to S$ as $N\to\infty$ ($\operatorname{diam}\Delta s_n\to 0$) (or if $S_\Delta\equiv S$, in the conformal finite element method [5]), then $(\mathbf{u}_N,\mathbf{t}_N^{(\nu_\Delta)})\to[\mathbf{u}_0,\mathbf{t}^{(\nu)}(\mathbf{u}_0)]$ in the sense that

$$|\mathbf{u}_N-\mathbf{u}_0|_{½,S}\to 0,\quad \|\mathbf{t}_N^{(\nu\Delta)'}-\mathbf{t}^{(\nu)}(\mathbf{u}_0)\|_{-½,S}\to 0 \tag{3.1}$$

where the norm $|\cdot|_{1/2,S}$ is defined in the subspace $W_2^{*1/2}(S)\subset W_2^{1/2}(S)$ on which the boundary quadratic form [6] $\langle\mathbf{u},\mathbf{t}^{(\nu)}(\mathbf{u})\rangle$ is positive definite and

$$|\mathbf{u}|_{½,S}=\{\langle\mathbf{u},\mathbf{t}^{(\nu)}(\mathbf{u})\rangle\}^{½}$$

For the proof, we use the variational equation (1.8) as an integral identity (with integrals over S_Δ) for $\mathbf{v}=\mathbf{u}-\mathbf{u}_0$, $\lambda=\lambda_0$ and for $\mathbf{v}=\mathbf{u}-\mathbf{u}_N$, $\lambda=\lambda_N$, obtaining two equalities. Noting that, by (1.10), when $\delta=\lambda_N$ and $\delta=\lambda_0$ we have $\mathbf{t}_{\lambda_N}^{(\nu_\Delta)}=\mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N)$ and $\mathbf{t}_{\lambda_0}^{(\nu)}=\mathbf{t}^{(\nu)}(\mathbf{u}_0)$, respectively, we set $\mathbf{u}=\mathbf{u}_N$, $\mathbf{t}_{\lambda_N}^{(\nu_\Delta)}=\mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N)$ in the first equality and $\mathbf{u}=\mathbf{u}_0$, $\mathbf{t}_{\lambda_0}^{(\nu)}=\mathbf{t}^{(\nu)}(\mathbf{u}_0)$, in the second, and then add the resulting equalities. This gives

$$w_\Delta(\mathbf{u}_0-\mathbf{u}_N,\ \mathbf{t}^{(\nu)}(\mathbf{u}_0)-\mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N))=\int_{S_\Delta}(\lambda_0-\lambda_N)[\mathbf{t}^{(\nu)}(\mathbf{u}_0)-\mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N)]\,ds_\Delta$$

If $S_\Delta\to S$ (or $S_\Delta\equiv S$), the right-hand side of this equality vanishes by (1.10) if $\delta=\lambda_0-\lambda_N$, and the left-hand side is equal to $|\mathbf{u}_0-\mathbf{u}_N|_{1/2,S}^2$. Consequently, the first convergence relation in (3.1) is indeed valid. Similarly, considering the variational equation (1.9), one can establish the second convergence relation in (3.1); alternatively, it follows from the estimate of the Trace theorem [5] if the first convergence is satisfied. The reader should note that this technique of proving convergence is to some extent standard; it has been used [13] to justify a duality algorithm for solving the generalized Signorini problem.

To estimate the error of the BEA, it is natural to use *a posteriori* error bounds, based on two-sided estimates of the functional $L(\mathbf{u}_0,\mathbf{t}^{(\nu)}(\mathbf{u}_0),\lambda_0)$. Such estimates turn out to be identical with those obtained previously [6] for coupled approximations.

In (2.1) we set $\mathbf{u}=\mathbf{u}_N$, $\mathbf{t}^{(\nu)}=\mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N)$, $\lambda=\lambda_N$ and form the difference of functionals

$$L_\Delta(\mathbf{u}_N,\mathbf{t}^{(\nu\Delta)}(\mathbf{u}_N),\lambda_0)-L_\Delta(\mathbf{u}_0,\mathbf{t}^{(\nu)}(\mathbf{u}_0),\lambda_N) \tag{3.2}$$

(the index Δ indicates expressions with integrals over S_Δ, and $\mathbf{u}_{0\Delta}$, $\lambda_{0\Delta}$ are the values of $\mathbf{u}_0$, λ_0 at points of S_Δ). Considering that the coupling equations (1.3) are satisfied by both exact and approximate solutions, so that $f_\Delta(\mathbf{u}_0,\ t\nu(\mathbf{u}_0),\ \lambda_N)=0$, $f_\Delta(\mathbf{u}_N,\ \mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N),\ \lambda_0)=0$ and using the expression (1.4) for the Lagrangian, we see that the difference (3.2) is equal to the difference of boundary functionals [see (1.1)]

$$F_{S_\Delta}(\mathbf{u}_N)-F_{S_\Delta}(\mathbf{u}_0)=w_\Delta(\mathbf{u}_N)-w_\Delta(\mathbf{u}_0)-2l(\mathbf{u}_N-\mathbf{u}_0) \tag{3.3}$$

where $w_\Delta(\mathbf{u}_N)=w_\Delta[\mathbf{u}_N,\ \mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N)]$, with a similar expression for $w_\Delta(\mathbf{u}_0)$.

We will now use the variational equation for $F_{S_\Delta}(\mathbf{u}_0)$ [see (1.1)], from which it follows that $l_\Delta(\mathbf{u}_N-\mathbf{u}_0)=w_\Delta(\mathbf{u}_0,\ \mathbf{u}_N-\mathbf{u}_0)$. Then the difference (3.3) may be written in the form

$$w_\Delta(\mathbf{u}_N) - 2w_\Delta(\mathbf{u}_0, \mathbf{u}_N) + w_\Delta(\mathbf{u}_0) = w_\Delta(\mathbf{u}_N - \mathbf{u}_0) = |\mathbf{u}_N - \mathbf{u}_0|^2_{½, S_\Delta}$$

To obtain an *a posteriori* bound, we now use a lower bound [6] for the functional $F_{S_\Delta}(\mathbf{u}_0)$ in terms of the functional $\Phi_{S_\Delta}[\mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N)]$ of the dual coupled problem. This gives

$$|\mathbf{u}_N - \mathbf{u}_0|^2_{½, S_\Delta} \leqslant F_{S_\Delta}(\mathbf{u}_N) - \Phi_{S_\Delta}(\mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N))$$

Here the right-hand side may be reduced to a form that is more convenient for calculations [6]

$$2 \int_{S_\Delta} \mathbf{u}_N [\mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N) - \mathbf{g}_N^{(\nu_\Delta)}] ds_\Delta$$

4. We will now consider some questions concerning applications. Previously proposed variational formulae of the BEM [1, 2] use the apparatus of discrete boundary potentials to approximate the solutions of the direct problem (in displacements) or the dual problem (in stresses), in which case these approximations are interdependent because of the defining relations (1.2) at the points of the discrete boundary. The formulations proposed here use a simultaneous uncoupled approximation of the solutions of the direct and dual problems. Thus, one can approximate the stress field at the points of the discrete boundary, independently of the field of displacements, in the following sense: if the problem setting involves making allowance for the increase of the stresses on some set of boundary points (e.g. at singular points of the contact of a rectangular-faced punch with a deformable medium), then the approximation of the boundary in the displacement field may be isoparametric, but the approximation in the stress field may be subparametric (with a large number of interpolation nodes). A similar approximation may be adopted to allow for the increase in stresses in the neighbourhood of the tips of a crack in crack problems (a variational formulation has been pointed out for the boundary functionals of these problems; see R. V. Gol'dshtein's appendix in [14]). The alternative, superparametric BEA is used in problems in which the approximation must make allowance for irregularities of the boundary. In both cases the number of nodes in the approximations of the boundary and the solution may be equal, but the order of system (2.4) depends on the number of approximation nodes of the solution, and the integral coefficients of the unknown nodal values of the displacements and the stresses are determined from basis functions of different orders.

Let us look more closely at the structure of the system of discrete boundary equations in the subparametric approximation; this is the case of interest in the applications mentioned above. Let $\{k\}_{1,\ldots,K}$, $\{k\}_{1,\ldots,K'}$ be the sets of interpolation nodes at the points of the BEs Δs_n ($n = 1, \ldots, N$), respectively, for interpolation of the displacement vectors $\mathbf{u}_n(\lambda_n)$ and stress vectors $\mathbf{t}_n^{(\nu_n)}$ from their nodal values $\mathbf{U}_{nk} = \{U_{nk}^{(i)}\}_{i=1,\ldots,m}$ and $\mathbf{T}_{nk} = \{T_{nk}^{(i)}\}_{i=1,\ldots,m}$; let ψ_k, $k \in \{k\}$, ψ'_k, $k \in \{k\}'$ be the corresponding basis functions of the BEM (in general of different orders in η). If $K \neq K'$, system (2.4) may be written in the form

$$\begin{gathered}
\sum_{n=1}^{N} \Big[\int_{U\Delta s_n} \sum_{i=1}^{m} \sum_{k,l=1}^{K'} T_{nk}^{(i)} \psi'_k \psi'_l |J_n| ds_n(\eta) + \int_{U\Delta s_n} \sum_{i=1}^{m} \sum_{k,l=1}^{K} \Lambda_{nk}^{(i)} \psi_k c \frac{\partial \psi_l}{\partial \nu_n} |J_n| ds_n(\eta) \Big] = \\
= 2 \sum_{n=1}^{N} \int_{U\Delta s_n} \sum_{i=1}^{m} \sum_{k,l=1}^{K'} Q_{nk}^{(i)} \psi'_k \psi'_l |J_n| ds_n(\eta) \\
\sum_{n=1}^{N} \Big[\int_{U\Delta s_n} \sum_{i=1}^{m} \sum_{k,l=1}^{K} U_{nk}^{(i)} \psi_k \psi_l |J_n| ds_n(\eta) - \int_{U\Delta s_n} \sum_{i=1}^{m} \sum_{k,l=1}^{K} \Lambda_{nk}^{(i)} \psi_k \psi_l |J_n| ds_n(\eta) \Big] = 0 \\
\sum_{n=1}^{N} \Big[\int_{U\Delta s_n} \sum_{i=1}^{m} \sum_{k,l=1}^{K'} T_{nk}^{(i)} \psi'_k \psi'_l |J_n| ds_n(\eta) - \int_{U\Delta s_n} \sum_{i=1}^{m} \sum_{k,l=1}^{K} U_{nk}^{(i)} c \frac{\partial \psi_k}{\partial \nu_n} \psi_l |J_n| ds_n(\eta) \Big] = 0
\end{gathered} \tag{4.1}$$

and the matrix of the system has variable band width. In the isoparametric approximation we have $\{k\} \equiv \{k\}'$, $\psi_k \equiv \psi'_k$, and system (4.1), as shown previously, reduces to (2.5). For the case $K' > K$ and $\{k\}' \supset \{k\}$ (i.e. additional interpolation nodes are considered for the stress field), using the equalities

$$\Lambda_{nk}^{(i)} = U_{nk}^{(i)}, \quad T_{nk}^{(i)} = c(\theta, \mu) U_{nk}^{(i)}, \quad \forall k \in \{k\} \tag{4.2}$$

(the second of which represents the validity of the defining relations for the nodal values of the displacements and stresses on the set $\{k\}$ of interpolation nodes of the Lagrange multipliers), we can reduce the solution of system (4.1) to the solution of a system of discrete boundary equations in the components of the nodal stresses $T_{nk}^{(i)}$

$$\sum_{n=1}^{N}\Big[\int_{U\Delta s_n}\sum_{i=1}^{m}\sum_{k,l=1}^{K'}T_{nk}^{(i)}\psi_k'\psi_l'|J_n|\,ds_n(\eta)+\int_{U\Delta s_n}\sum_{i=1}^{m}\sum_{k,l=1}^{K}T_{nk}^{(i)}\psi_k\frac{\partial\psi_l}{\partial\nu_n}|J_n|\,ds_n(\eta)\Big]=$$
$$=2\sum_{n=1}^{N}\int_{U\Delta s_n}\sum_{i=1}^{m}\sum_{k,l=1}^{K'}Q_{nk}^{(i)}\psi_k'\psi_l'|J_n|\,ds_n(\eta)\qquad(4.3)$$

where $Q_{nk}^{(i)}$ are the nodal values of the components of the stress vector $\mathbf{g}^{(\nu)}$ [see (1.1)].

The justification of the *a priori* given defining relations (4.2) follows from the fact that if the first term of system (4.3) is substituted into the third equation of system (4.1), and allowance is made for (4.2) and the symmetry of the coefficients of the boundary element system (see above), we obtain a system whose right-hand side includes the contributions of the nodes $k\in\{k\}'$; restriction of the right-hand side to the set of nodes $\{k\}$ yields a system corresponding to system (2.5) for a coupled BEA, whose solution implements the defining relations (4.2). After using (4.3) to determine the values of $T_{nk}^{(i)}$ on the set $\{k\}'$, the values of $U_{nk}^{(i)}$, $i=1,\ldots,m$ on $\{k\}$ are determined using the third equation of system (4.1).

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Translated by D.L.